1 The Rise of Stromgren Spheres

1a) Defining $\dot{N}_{\rm ion}$ ($\dot{N}_{\rm rec}$) as the total number of photoionizations (recombinations) per second that occur in the surrounding gas, we have the expression

$$\frac{dN_{\rm tot}}{dt} = \dot{N}_{\rm ion} - \dot{N}_{\rm rec} \tag{1}$$

As in the standard Stromgren argument, we say that all ionizing photons emitted by the star are absorbed in the surrounding medium, thus $\dot{N}_{\rm ion}$ is simply the number of ionizing photons emitted per second, denoted by Q

$$\dot{N}_{\rm ion} = Q = \int_0^\infty \frac{L_\nu(\nu)}{h\nu} d\nu \tag{2}$$

Since we assume that all photons come out at a single frequency ν_0, we simply have

$$Q = \frac{L}{h\nu_0} \tag{3}$$

The number of total radiative recombinations per second is

$$\dot{N}_{\rm rec} = n_e n_p \alpha_B V \tag{4}$$

where n_e, n_p are the electron and proton number densities, V is the volume, and we will use the α_B recombination coefficient (i.e., we will assume all ionizing photons are trapped in the HII region). Finally, the total number of atoms in the HII region is $N_{\rm tot} = nV$. Assuming the HII region is essentially totally ionized, $n_p = n_e = n$, where the total number density n is assumed to be constant with radius. Then equation 1 can be written

$$n\frac{dV}{dt} = Q - n^2\alpha_B V \tag{5}$$

which can be written

$$\frac{dV}{dt} = n\alpha_B\left[\frac{Q}{n^2\alpha_B} - V\right] \tag{6}$$

We see the characteristic scales are the recombination time $t_{\rm rec} = (n\alpha_B)^{-1}$ and the Stromgren volume, $V_s = 4\pi R_s^3/3 = Q/n^2\alpha_B$, where the stromgren radius is

$$R_s = \left[\frac{3Q}{4\pi n^2\alpha_B}\right]^{1/3} \tag{7}$$

So a nice way to write the differential equation is

$$\frac{dV}{dt} = -\frac{V_s}{t_{\rm rec}}\left[\frac{V}{V_s} - 1\right] \tag{8}$$

The solution is

$$\ln(V/V_s - 1) = -t/t_{\rm rec} + C \tag{9}$$

The integration constant $C = 0$ since the volume is zero at $t = 0$, so

$$V(t) = V_s(1 - e^{-t/t_{\rm rec}}) \tag{10}$$

or in terms of the radius

$$R(t) = R_s(1 - e^{-t/t_{\rm rec}})^{1/3} \tag{11}$$

We see that as $t \to \infty$, the radius of the HII region goes to R_s, as expected. The timescale for the HII region to grow is given by the recombination timescale $t_{\rm rec}$.

1b) At a temperature $T = 10^4$ K, the recombination coefficient is $\alpha_B \approx 2 \times 10^{-13}$. So plugging in numbers we find

$$t_{\rm rec} = (n\alpha_B)^{-1} \approx 1.6 \times 10^5 \text{ years} \tag{12}$$

This is comparable to the lifetime of an O-star, so the HII region will just about grow to its stromgren radius when the star is about to die.

1c) We can find the velocity of the edge of the HII region by differentiating our solution

$$v(t) = \frac{dR(t)}{dt} = \frac{R_s}{t_{\rm rec}} \frac{e^{-t/t_{\rm red}}}{3} (1 - e^{-t/t_{\rm rec}})^{-2/3} \tag{13}$$

We could have guessed that characteristic velocity of the HII region expansion is

$$v_s \sim R_s/t_{\rm rec} \approx 10^7 \text{ cm s}^{-1} \tag{14}$$

This is about an order of magnitude greater than the sound speed $c \sim (kT/m_p)^{1/2} \sim 10^6$ cm s^{-1}. Thus the HII region expansion is initially supersonic and we can neglect hydrodynamical effects.

1d) To determine the ionization state at a radius r, we apply the equation that expresses photoionization equilibrium

$$4\pi \int_0^\infty \frac{J_\nu(r)}{h\nu} \sigma(\nu) n_{HI} = n_e n_p \alpha_B \tag{15}$$

which states that the local photoionization rate equals the radiative recombination rate. Here $\sigma(\nu)$ is the bound-free cross-section for hydrogen, and $J_\nu(r)$ the monochromatic mean intensity of the radiation field at radius r. Assuming the radiation source is an isotropically emitting point source of intensity I_ν and radius $R_\star$, and there is negligible attenuation above it (which should be OK at very small radii) we can use the standard result

$$J_\nu = \frac{I_\nu R_\star^2}{4r^2} \tag{16}$$

The monochromatic flux at the surface of a lamber radiator is $F_\nu = \pi I_\nu$ and so the monochromatic luminosity is

$$L_\nu = 4\pi R_\star^2 F_\nu = 4\pi^2 R_\star^2 I_\nu \tag{17}$$

So we can write the monochromatic mean intensity as

$$J_\nu = \frac{L_\nu}{16\pi^2 r^2} \tag{18}$$

(Note that the original problem set neglected a factor of 4π). Using this value, the photoionization equilibrium equation becomes

$$\frac{Q\sigma_0}{4\pi r^2} n_{\rm HI} = n_e n_p \alpha_B \tag{19}$$

where $\sigma_0 = \sigma(\nu_0)$. Defining the ionized fraction $x_{\rm HII} = n_{HII}/n$, we have for pure hydrogen, $n_e = n_p = x_{\rm HII} n$ and $n_{\rm HI} = (1 - x_{\rm HII})n$ and so

$$\frac{4\pi Q n}{r^2}(1 - x_{\rm HII}) = n^2 x_{\rm HII}^2 \alpha_B \tag{20}$$

Dividing both sides by $n^2\alpha_B$, we can write this in dimensionless form

$$2\Omega(1 - x_{\rm HII}) = x_{\rm HII}^2 \quad \text{where } \Omega = \frac{2\pi Q\sigma_0}{nr^2\alpha_B} \tag{21}$$

Where the dimensionless quantity Ω is apparently the useful measure for how strong the ionization is. We can now solve the quadratic equation for $x_{\rm HII}$

$$x_{\rm HII} = -\Omega + \sqrt{\Omega^2 + 2\Omega} \tag{22}$$

and hence

$$x_{\rm HI} = 1 - x_{\rm HII} = 1 + \Omega - \sqrt{\Omega^2 + 2\Omega} \tag{23}$$

We should check our limits. For a very weak source ($L \to 0$ at fixed r), we have $\Omega \to 0$ and find no ionization $x_{\rm HII} = 0$, as expected. For a very strong source ($L \to \infty$ at fixed r), we have $\Omega \to \infty$, and we need to take a little more care in taking the limit. We use a Taylor expansion in the small quantity $2/\Omega$:

$$x_{\rm HII} = -\Omega + \Omega(1 + 2/\Omega)^{1/2} \approx -\Omega + \Omega(1 + 1/\Omega + ...) \approx 1 \tag{24}$$

As expected, the medium is totally ionized.

1e) To find the small r behavior, we note that this is the limit $\Omega \to \infty$, so we use an Taylor expansion similar to the above, but keep another term

$$x_{\rm HI} = 1 + \Omega - \Omega(1 + 2/\Omega)^{1/2} \approx 1 + \Omega - \Omega(1 + 1/\Omega - 1/4\Omega^2 + ...) \tag{25}$$

Thus to lowest order in $1/\Omega$ we find

$$x_{\rm HI} \approx \frac{1}{4\Omega} \propto \frac{r^2 n}{L} \tag{26}$$

2 Flipping Spins at the Epoch of Reionization

Define the excitation energy for the 21 cm line to be $T_\star = h\nu_{\rm fs}/k = 0.068$ K, where $\nu_{\rm fs} = (c/21 \text{ cm})$. We are always in the Rayleigh-Jeans limit since $T \gg T_\star$ so the observed specific intensity is

$$I_{\nu,\rm obs} = \frac{2\nu_{fs}^2}{c^2} kT_b, \tag{27}$$

where T_b is the brightness temperature, which may or may not be related to the kinetic temperature of the observed gas, T_K. Lastly, the spin temperature T_s is defined as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-T_\star/T_s} = 3e^{-T_\star/T_s}. \tag{28}$$

a) Consider a CMB beam with specific intensity $B_\nu(T_\gamma, \nu_{fs}) \equiv I_0$ passing through a neutral hydrogen cloud of optical depth τ. The radiative transfer equation tells us that

$$\frac{\partial I_\nu}{\partial \tau} = -I_\nu + S_\nu. \tag{29}$$

If S_ν is constant in space, we can let $x = I_\nu - S_\nu$ and the solution to the above equation is

$$x = x_0 e^{-\tau}. \tag{30}$$

Substituting back in for x, we find that the observed specific intensity for the CMB beam is:

$$I_{\nu,\rm obs} = B_\nu(T_\gamma, \nu_{\rm fs})e^{-\tau} + S_\nu(\nu_{\rm fs})(1 - e^{-\tau}). \tag{31}$$

Now, we must calculate the source function of the 21 cm line, $S_\nu = j_\nu/\alpha_\nu$. In terms of the Einstein coefficients, the extinction coefficient corrected for stimulated emission is

$$\alpha_\nu = \frac{h\nu_{\rm fs}}{4\pi}\phi(\nu_{\rm fs})n_0 B_{01}\left(1 - \frac{n_1 B_{10}}{n_0 B_{01}}\right). \tag{32}$$

Now use the Einstein relation $B_{10}/B_{01} = g_0/g_1$ and substitute in for n_1/n_0:

$$\alpha_\nu = \frac{h\nu_{\rm fs}}{4\pi}\phi(\nu_{\rm fs})n_0 B_{01}\left(1 - e^{-T_\star/T_s}\right). \tag{33}$$

Now use another Einstein relation $A_{10} = (2h\nu_{\rm fs}^3/c^2)B_{10} = (2h\nu_{\rm fs}^3/3c^2)B_{01}$,

$$\alpha_\nu = \frac{3c^2}{8\pi\nu_{\rm fs}^2}\phi(\nu_{\rm fs})n_0 A_{10}\left(1 - e^{-T_\star/T_s}\right). \tag{34}$$

Lastly, we can Taylor expand the exponential since $T_s \gg T_\star$ to obtain the extinction coefficient for the 21 cm line:

$$\alpha_\nu = \frac{3c^2}{8\pi\nu_{\rm fs}^2}\phi(\nu_{\rm fs})n_0 A_{10}\left(\frac{T_\star}{T_s}\right). \tag{35}$$

We can also express the emission coefficient in terms of the Einstein coefficients:

$$j_\nu = \frac{h\nu_{\rm fs}}{4\pi}\phi(\nu_{\rm fs})n_1 A_{10}. \tag{36}$$

Now we can compute the source function by dividing the emission and extinction coefficients:

$$S_\nu = \frac{2h\nu_{\rm fs}^3}{3c^2}\frac{n_1}{n_0}\frac{T_s}{T_\star} = \frac{2h\nu_{\rm fs}^3}{c^2}\frac{T_s}{T_\star}e^{-T_\star/T_s}. \tag{37}$$

In the limit of $T_s \gg T_\star$, we obtain the final result for the source function

$$S_\nu = \frac{2\nu_{\rm fs}^2}{c^2}kT_s. \tag{38}$$

Now return to equation 31 and plug in for the source function:

$$I_{\nu,\rm obs} = B_\nu(T_\gamma, \nu_{\rm fs})e^{-\tau} + \frac{2\nu_{\rm fs}^2}{c^2}kT_s(1 - e^{-\tau}). \tag{39}$$

Plugging in for $I_{\nu,\rm obs}$, we find the brightness temperature to be:

$$T_b = T_\gamma e^{-\tau} + T_s(1 - e^{-\tau}). \tag{40}$$

Relative to the CMB background temperature, we have $\delta T_b \equiv T_b - T_\gamma$, or

$$\boxed{\delta T_b = (T_s - T_\gamma)(1 - e^{-\tau})}. \tag{41}$$

b) In statistical equilibrium, the balance of Einstein coefficients is

$$n_1(A_{10} + B_{10}\bar{J} + C_{10}) = n_0(B_{01}\bar{J} + C_{01}), \tag{42}$$

or

$$\frac{n_1}{n_0} = \frac{B_{01}\bar{J} + C_{01}}{A_{10} + B_{10}\bar{J} + C_{10}}. \tag{43}$$

Use the following three Einstein relations:

$$\begin{aligned} B_{01} &= 3B_{10}, \\ A_{10} &= 2h\nu_{\rm fs}^3 B_{10}/c^2, \\ C_{01} &= 3C_{10}e^{-T_\star/T_K}. \end{aligned}$$

Also, the local mean intensity field $\bar{J}$ is defined such that

$$\bar{J} = \frac{2\nu_{\rm fs}^2}{c^2} kT_\gamma. \tag{44}$$

Using the above 4 relations, we can express equation 43 in terms of C_{10} and A_{10}. Defining $x_c = C_{10}T_\star/A_{10}T_\gamma$, we obtain the desired result:

$$T_s^{-1} = \frac{T_\gamma^{-1} + x_c T_K^{-1}}{1 + x_c}. \tag{45}$$

From inspection of the above equation we see that $\boxed{T_s = T_K \text{ when } x_c \gg 1}$ and $\boxed{T_s = T_\gamma \text{ when } x_c \ll 1}$. When $x_c \gg 1$, collisional de-excitation is dominant so the gas collisions are setting the spin-flip level populations and the spin temperature will approach the gas temperature. In the opposite limit, the level populations are being set by the radiative transitions caused by the CMB so the spin temperature approaches T_γ in this limit.

c) We want to find the critical density such that $x_c = 1$. For collisional de-excitation by neutral hydrogen collisions, the rate is roughly given by

$$C_{10} \simeq \sigma_{10} n_H v_H. \tag{46}$$

We will assume the cross section is just the geometric cross section, $\sigma_{10} = \pi a_0^2$, where a_0 is the Bohr radius. Assume a characteristic velocity of order

$$v_H \simeq \sqrt{\frac{kT_K}{m_H}}. \tag{47}$$

Plugging in we find the critical density to be:

$$n_{H,c} = \frac{A_{10}T_\gamma}{\pi a_0^2 T_\star}\sqrt{\frac{m_H}{kT_K}}. \tag{48}$$

In class we estimated the spontaneous emission coefficient for hyperfine splitting to be $A_{10} \simeq 6\times10^{-15}\ \text{s}^{-1}$. Using $T_K = 100$ K and $T_\gamma = 2.7$ K, the critical density is roughly $\boxed{n_{H,c} \simeq 3 \times 10^{-2}\ \text{cm}^{-3}}$.

d) The rate at which Lyα photons drive transitions from the excited to the ground hyperfine level is given by P_{10}. By LTE arguments (similar to the derivation for the relation between C_{10} and C_{01} from class), we have

$$P_{01} = 3P_{10}e^{-T_\star/T_K}. \tag{49}$$

Statistical equilibrium now including transitions due to Lyα photons is given by

$$n_1(A_{10} + B_{10}\bar{J} + C_{10} + P_{10}) = n_0(B_{01}\bar{J} + C_{01} + P_{01}). \tag{50}$$

Analogous to part b), we can use the Einstein relations and the relation for $\bar{J}$ to express the above equation only in terms of A_{10}, C_{10}, and P_{10}. The relation we obtain is

$$T_s^{-1} = \frac{T_\gamma^{-1} + T_K^{-1}(x_c + x_\alpha)}{1 + x_c + x_\alpha}, \tag{51}$$

where

$$x_\alpha = \frac{P_{10}}{A_{10}}\frac{T_\star}{T_\gamma}. \tag{52}$$

e) From part a), the observed fluctuation in the brightness temperature is given by equation 40. Absorption occurs when $\delta T_b < 0$ and emission occurs when $\delta T_b > 0$. From inspection of equation 40, the sign of δT_b depends on the quantity $T_s - T_\gamma$, so we must determine the sign of this quantity as a function of redshift.

(a) $200 \leq z \leq 1100$

- $n_H > n_{H,c}$ so we are in the limit of $x_c \gg 1$
- we know from part b) that $T_s = T_K$ in this limit
- the gas and CMB are still thermally coupled so $T_K = T_\gamma$
- so we have $T_s = T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal

(b) $40 \leq z \leq 200$

- $n_H > n_{H,c}$ still so we are in the limit of $x_c \gg 1$ and $T_s = T_K$
- the gas falls out of equilibrium with the CMB and cools adiabatically
- for a monotonic adiabatic gas, with no heating sources:

$$TV^{2/3} = constant. \tag{53}$$

  The volume scales with redshift as $(1+z)^{-3}$ so we see that

$$T_K \propto (1+z)^2. \tag{54}$$

  The CMB temperature scales with redshift as $T_\gamma \propto 1+z$. Thus, we see that once the gas falls out of equilibrium with the CMB, the gas will cool faster and $T_K < T_\gamma$.
- So we have:

$$T_s - T_\gamma = T_K - T_\gamma < 0, \tag{55}$$

  and we see that $\boxed{\delta T_b < 0}$ $\rightarrow$ absorption.

(c) $30 \leq z \leq 40$

- now $n_H < n_{H,c}$ so we are in the limit of $x_c \ll 1$
- we know from part b) that $T_s = T_\gamma$ in this limit
- so we have $T_s - T_\gamma = T_\gamma - T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal

(d) $15 \leq z \leq 30$

- the production of Lyα photons leads to Lyα scattering setting the spin-flip level populations so $x_\alpha \gg 1$
- we know from part d) that $T_s = T_K$ in this limit
- due to adiabatic cooling and a lack of heating, $T_K < T_\gamma$
- so we have $T_s - T_\gamma = T_K - T_\gamma < 0$ and $\boxed{\delta T_b < 0}$ $\rightarrow$ absorption

(e) $7 \leq z \leq 15$

- the production of Lyα photons leads to Lyα scattering setting the spin-flip level populations so $x_\alpha \gg 1$
- we know from part d) that $T_s = T_K$ in this limit
- first sources heat the gas such that now $T_K > T_\gamma$
- so we have $T_s - T_\gamma = T_K - T_\gamma > 0$ and $\boxed{\delta T_b > 0}$ $\rightarrow$ emission

(f) $z \leq 7$

- reionization has ionized the universe so neutral hydrogen is negligible
- thus, $x_c = x_\alpha = 0$ and $T_s = T_\gamma$
- so we have $T_s - T_\gamma = T_\gamma - T_\gamma$ and $\boxed{\delta T_b = 0}$ $\rightarrow$ no signal